

NONCOMMUTATIVE PHYSICS ON LIE ALGEBRAS, $(\mathbb{Z}_2)^n$ LATTICES AND CLIFFORD ALGEBRAS

S. MAJID

ABSTRACT. We survey noncommutative spacetimes with coordinates being enveloping algebras of Lie algebras. We also explain how to do differential geometry on noncommutative spaces that are obtained from commutative ones via a Moyal-product type cocycle twist, such as the noncommutative torus, θ -spaces and Clifford algebras. The latter are noncommutative deformations of the finite lattice $(\mathbb{Z}_2)^n$ and we compute their noncommutative de Rham cohomology and moduli of solutions of Maxwell's equations. We exactly quantize noncommutative $U(1)$ -Yang-Mills theory on $\mathbb{Z}_2 \times \mathbb{Z}_2$ in a path integral approach.

1. INTRODUCTION

Noncommutative geometry has made rapid progress in the past two decades and is arguably now mature enough to be fully computable and relevant to real physical experiments. The benefit of noncommutative geometry is that whereas the usual coordinates on a space are commutative and most ‘nice’ commutative algebras could be viewed that way, noncommutative geometry, by relaxing the commutativity assumption, allows practically any algebra to be viewed geometrically. This opens up a brave new world of possibilities for model-building in which our favorite noncommutative algebras, such as matrices, angular momentum operators, Clifford algebras, can be viewed as noncommutative coordinates with all of the geometry that that entails. And why should we need such a possibility? First of all, we already encounter such noncommutative geometries when we quantise a system. Recall that classical mechanics is described by classical symplectic geometry, Hamilton-Jacobi equations of motion, etc. What happens to all this geometry when we quantise, is it all just thrown away after we have got the right commutators? The correspondence principle in quantum mechanics says that it should not be thrown away, that classical variables such as angular momentum should have their analogues in the quantum system with analogous properties. But this is a rather vague statement. Analogues of what? It seems likely that the quantum system should be as rich if not richer in structure than the classical one and should have among other things analogues of all of the geometry of phase space, Hamiltonian flows etc. if only we had the ‘eyes’ to see it. Noncommutative geometry can in principle provide those eyes.

Date: 2/2003.

1991 Mathematics Subject Classification. 58B32, 58B34, 20C05, 81T75.

Key words and phrases. Quantum groups, noncommutative geometry, quantum field theory, quantum gravity, finite lattice, twisting.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In this article we will not apply noncommutative geometry to quantum phase space exactly, though that is an interesting direction that deserves more study. Instead we will look at the even more radical proposal that spacetime itself may have noncommutative coordinates, an as yet undiscovered but potentially new physical phenomenon if it were ever verified. It could be called ‘cogravity’ for reasons explained in Section 2. The idea is not new [1], but we shall focus on three simple examples based on Lie algebras viewed as noncommutative spacetimes, with some new observations for the θ -spacetime case.

After the preliminary Section 2, we shall move to new results. The first, in Section 3 is an application of noncommutative geometry to quantize $U(1)$ -Yang-Mills theory on the finite set $\mathbb{Z}_2 \times \mathbb{Z}_2$ all the way up to Wilson loop vacuum expectation values. Section 4 covers the electromagnetic theory on Clifford algebras $\text{Cliff}(n)$ viewed as noncommutative coordinates. This is not only for fun; Clifford algebras can be viewed as ‘Moyal-product’ cocycle type quantizations of \mathbb{Z}_2^n in the same spirit as recently popular proposals for θ -spacetime in string theory, but now in a discrete version. Section 5 is a more mathematical section for experts and is the general theory behind such cocycle twist quantisations. This theory depends on Drinfeld’s quantum groups work but its full impact remains not so well-known in the operator theory approach to noncommutative geometry of Connes [2] and others. We point out that cocycle twisting means by the Majid-Oeckl twisting theorem[3] that the differential geometry also twists (or gets ‘quantised’) by the same cocycle and that this covers the algebraic (but not functional analytic) aspects of many famous examples including the celebrated noncommutative torus A_θ .

Next, a few words about the relation between the operator theory and quantum groups approaches to noncommutative geometry. From a mathematical point of view the former began in the 1940’s with the theorem of Gelfand and Naimark characterising functions on locally compact spaces as commutative C^* algebras and hence proposing any noncommutative C^* algebra as a ‘noncommutative space’. Vector bundles were characterised in the 1960’s by the Serre-Swann theorem as finitely generated projective modules, a notion again working in the noncommutative case. Operator K-theory led in the 1980’s to cyclic cohomology and ultimately to Connes formalisation of the Dirac operator on a spin manifold in operator algebra terms as a ‘spectral triple’[2]. At the same time in the 1980’s there emerged from the theory of integrable systems a large class of examples of noncommutative algebras with manifest ‘geometrical’ content, namely quantum groups $C_q[G]$ deforming the usual coordinate ring of a simple Lie group G . Also at the same time in the 1980s there emerged a different ‘bicrossproduct’ class of quantum groups [4] again built on Lie groups but this time from a self-duality approach to quantum gravity and Planck-scale physics. Moreover, just as Lie groups and their homogeneous spaces provide key examples of classical differential geometry, so quantum groups *should* provide key examples of noncommutative geometry. This ‘quantum groups approach’ to noncommutative geometry starts with differential structures on quantum groups [5], gauge theory with quantum group fiber[6] and eventually a notion of a ‘quantum manifold’ as an algebra equipped with a quantum group frame bundle [7]. It is important for us here, however, that whereas the q -deformation literature and examples like $C_q[SU_2]$ played an important role in developing the right axioms, the final theory, like the operator algebras approach, applies in principle to any algebra including ones that have nothing whatever to do with q -deformations. It should be said also that the operator theory approach tends to be mathematically

more sophisticated while the quantum groups one tends to be more computational and examples-led. One of the aims of Section 5 is to promote convergence between the two approaches.

We will need some precise notations. Let M be a unital algebra over \mathbb{C} (say). We regard M as ‘coordinate algebra’ on a space, even though it need not be commutative. By ‘differential calculus’ we mean (Ω^1, d) where Ω^1 is an $M - M$ bimodule, $d : M \rightarrow \Omega^1$ obeys the Leibniz rule

$$(1) \quad d(fg) = (df)g + fdg, \quad \forall f, g \in M$$

and elements of the form fdg span Ω^1 . We also want in practice that the kernel of d is spanned by 1, the constants. Note that we only need Ω^1 to be a bimodule – indeed, assuming $fdg = (dg)f$ for all f, g would imply $d(fg - gf) = 0$ which would make d trivial on a very noncommutative algebra. So we do not assume this, but only the weaker bimodule condition $f((dg)h) = (fdg)h$ for all f, g, h . This is the starting point of all approaches to noncommutative geometry. After this, one can extend to an entire exterior algebra of differential forms Ω, d with d a super-derivation of degree 1 with $d^2 = 0$. How this is done is different in the two approaches; in the quantum groups one we use considerations of a background symmetry to narrow down the possible extensions with the result that in most cases there is a reasonably natural choice. In the Connes approach it is defined by choosing an operator to be the ‘Dirac’ operator. Another ingredient one needs is a Hodge \star operator $\Omega^m \rightarrow \Omega^{n-m}$ where n is the top degree (which we assume exists, and which is called the volume-dimension of the noncommutative space).

With these few ingredients one can already do a lot of physics, in particular most of electromagnetism. The simplest version of this which we call ‘Maxwell theory’ is to think of a gauge field $A \in \Omega^1$ modulo exact forms, with curvature $F = dA$. Next there is a nonlinear version which we call $U(1)$ -Yang-Mills theory where we consider $A \in \Omega^1$ modulo the transformation

$$(2) \quad A \mapsto uAu^{-1} + udu^{-1}$$

with u an invertible element of M . Here the curvature is $F = dA + A \wedge A$ and transforms by conjugation. Such a theory has all the flavour of a nonAbelian gauge theory but not because of a nonAbelian gauge group, rather because differentials and functions do not commute. One can go on and do nonAbelian gauge theory, Riemannian geometry and construct a Dirac operator \not{D} . This would be beyond our present scope but we refer to concrete models such as in [7][8][9]. It should be mentioned that the results do not usually fit precisely the axioms of a spectral triple, but perhaps something like them.

2. NONCOMMUTATIVE SPACETIME AND COGRAVITY

This section is by way of ‘warm up’ for the reader to get comfortable with the methods in a more familiar setting before applying it to other algebras. We look particularly at models of noncommutative spacetime where the coordinate algebra is the enveloping algebra $U(\mathfrak{g})$ and \mathfrak{g} is a Lie algebra, with structure constants c_{ij}^k say. Putting in a parameter λ , it means commutation relations

$$(3) \quad [x_i, x_j] = \lambda c_{ij}^k x_k$$

This is a well-studied system in mathematical physics, namely $U(\mathfrak{g})$ is the standard quantisation of the Kirillov-Kostant Poisson bracket on \mathfrak{g}^* defined canonically by

the Lie algebra structure. The symplectic leaves for this are the coadjoint orbits and their quantisation is given by setting the Casimir of $U(\mathfrak{g})$ to a fixed number.

What if actual space or spacetime coordinates x_i, t might be elements of such a noncommutative algebra instead of numbers? As in quantum mechanics it means of course that they cannot be simultaneously diagonalised so that there will be some uncertainty or order-of-measurement dependence in the precise location of an event. But there would be many more effects as well, depending on the noncommutative algebra used. We nevertheless propose:

Claim 2.1. (A) *Noncommutative x_i, t would be a new (as yet undiscovered) physical effect ‘cogravity’*

(B) *Even if originating in quantum gravity corrections of Planck scale order, the effect could in principle be tested by experiments today.*

We will look at the second claim in Section 2.1 by way of our first example, the Majid-Ruegg κ -spacetime. Here we look briefly at claim (A). The first thing to note is that such spacetimes $U(\mathfrak{g})$ are noncommutative analogues of *flat* space. This is because they are Hopf algebras with a trivial additive coproduct and counit

$$(4) \quad \Delta 1 = 1 \otimes 1, \quad \epsilon 1 = 1, \quad \Delta \xi = \xi \otimes 1 + 1 \otimes \xi, \quad \epsilon \xi = 0, \quad \forall \xi \in \mathfrak{g}$$

extended multiplicatively, which is to say an additive abelian ‘group’ structure on the noncommutative space. On the other hand, if these were coordinates of spacetime then the dual Hopf algebra should be the momentum coordinate algebra. Here it is the usual commutative coordinate ring $\mathbb{C}[G]$ of the underlying Lie group G , which has a nonAbelian multiplicative coproduct for group multiplication. So momentum space is a nonAbelian group manifold with curvature (at least in compact cases like $SU_2 = S^3$), but an ordinary (commutative) space. Moreover, there is a nonAbelian Fourier transform

$$(5) \quad \mathcal{F} : U(\mathfrak{g}) \rightarrow \mathbb{C}[G]$$

or more precisely on a completion of these algebras to allow functions with good decay properties. We see that the physical meaning of noncommutative space coordinates is equivalent under Fourier transform to curvature or gravity in momentum space. The reader is probably more familiar with the flipped situation in which position space is classical and curved, such as a nonAbelian group $SU(2)$, and its Fourier dual is a noncommutative space with noncommuting momenta or covariant derivatives, such as $[p_i, p_j] = i\epsilon_{ijk}p_k$. The idea of noncommutative space or spacetime is mathematically just the same but the with roles of position and momentum swapped. We summarise a typical situation in Table 1 to give some idea of the physical meaning of the mathematics that we propose here. It makes it clear that this reversed possibility of noncommutative position space and curved momentum space is a potentially new physical effect dual to gravity. Associated to it is a new dimensionful parameter, say λ controlling the noncommutativity of spacetime or curvature of momentum space. It is independent of usual curvature in position space, which is associated to the Newton constant parameter, and to Planck’s constant which controls noncommutativity between the position and momentum sectors. This point of view was explained in detail in [1].

Turning to technical details, we use the following general construction which comes out of the analysis of translation-invariant differential structures on quantum groups. If $M = U(\mathfrak{g})$, a differential calculus is specified by a right ideal $\mathcal{I} \subset U(\mathfrak{g})^+$

	Position	Momentum
Gravity	Curved $\sum_i x_i^2 = 1$	Noncommutative $[p_i, p_j] = \imath \epsilon_{ijk} p_k$
Cogravity	Noncommutative $[x_i, x_j] = \imath \epsilon_{ijk} x_k$	Curved $\sum_i p_i^2 = 1$
Qua.Mech.	$[x_i, p_j] = \imath \hbar \delta_{ij}$	

FIGURE 1. Gravity and cogravity related by nonAbelian Fourier transform

where $U(\mathfrak{g})^+$ denotes expressions in the generators with no constant term. Let $\Lambda^1 = U(\mathfrak{g})^+/\mathcal{I}$ be the quotient and $\pi : U(\mathfrak{g})^+ \rightarrow \Lambda^1$ be the projection map. Let $\tilde{\pi}$ be the projection from $U(\mathfrak{g})$ where we first apply $\text{id} - 1\epsilon$ which projects to $U(\mathfrak{g})^+$, and then π . We have

$$(6) \quad \Omega^1 = U(\mathfrak{g})\Lambda^1, \quad df = (\text{id} \otimes \tilde{\pi})\Delta f, \quad \omega f = f_{(1)}\pi(\tilde{\omega}.f_{(2)}), \quad \forall f \in U(\mathfrak{g}), \omega \in \Lambda^1,$$

where $\tilde{\omega}$ is a representative projecting onto ω and $\Delta f = f_{(1)} \otimes f_{(2)}$ is a notation. The elements of Λ^1 are the ‘basic 1-forms’ and others are given by these with ‘functional’ coefficients from $U(\mathfrak{g})$ on the left. Note that 1-forms and such ‘functions’ do not commute. The wedge product for the entire exterior algebra is given in the present class of examples simply by elements of Λ^1 anticommuting among themselves. The Hodge \star operator is similarly the usual one among such basic one-forms. Finally, given a basis $\{e_\mu\}$ of Λ^1 , we define partial derivatives ∂^μ by (summation understood):

$$(7) \quad df = (\partial^\mu f)e_\mu, \quad \forall f \in U(\mathfrak{g}).$$

A natural way to specify the ideal \mathcal{I} is as the kernel of a representation $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ extended as a representation of $U(\mathfrak{g})^+$. Then we can identify Λ^1 as the image of ρ and have the formulae

$$(8) \quad d\xi = \rho(\xi), \quad [\omega, \xi] = \omega.\rho(\xi), \quad \forall \xi \in \mathfrak{g}, \omega \in \text{im}(\rho),$$

where we take the matrix product on the right.

2.1. λ -spacetime and gamma-ray bursts. We start with one of the most accessible noncommutative spacetime in this family[11]

$$(9) \quad [t, x_i] = \imath \lambda x_i, \quad [x_i, x_j] = 0$$

where λ has time dimension. If the effect is generated by quantum gravity corrections, we might expect $\lambda \sim 10^{-44}s$, the Planck time. One can also work with $\kappa = \lambda^{-1}$. Such a spacetime in two dimensions (i.e. the enveloping algebra of $U(b_+)$ where $b_+ \subset su_2$) was first proposed in [10] with an additional q parameter which one may set to 1. The first thing the reader will be concerned about is that this proposal manifestly breaks Lorentz invariance, so cannot be correct. What was shown in [11] that there is not only Lorentz but a full Poincaré invariance, but under a quantum group

$$(10) \quad U(so_{1,3}) \bowtie \mathbb{C}[\mathbb{R}^{1,3}].$$

This was shown also to be isomorphic to a ‘ κ -Poincaré’ quantum group that had been proposed from another point of view (that of contraction from $U_q(so_{2,3})$) by

J. Lukierski et al. in [12] but without a noncommutative spacetime on which to act.

What is important is that all of the geometrical consequences are not ad hoc but naturally follow within our approach to noncommutative geometry from the choice (9) of algebra. The first step is to compute the natural differential structure. There is not much choice and we take the 4-dimensional representation

$$(11) \quad \rho(x_\mu) = \imath\lambda \begin{pmatrix} 0 \\ e_\mu \end{pmatrix}$$

where the $e_\mu = (0 \cdots 1 \cdots 0)$ has 1 in the $\mu + 1$ -th position and $\mu = 0, 1, 2, 3$. We write $x_0 \equiv t$. The basic 1-forms are provided by the dx_μ as certain matrices, and these span the image of ρ since the exponentiation of the Lie algebra has the similar form in this representation. The result from the general theory above is then

$$(12) \quad (dx_j)x_\mu = x_\mu dx_j, \quad (dt)x_\mu - x_\mu dt = \imath\lambda dx_\mu.$$

The form for d then implies

$$(13) \quad \partial^i : f(x, t) :=: \partial \partial x_i f(x, t) :, \quad \partial^0 : f(x, t) :=: f(x, t + \imath\lambda) - f(x, t)\imath\lambda :$$

for normal ordered polynomial functions. We use such normal ordered functions, with t to the right, to describe a general function in the spacetime. Under this identification we can extend all formulae to formal power-series. Note that we see the effect of the noncommutative spacetime as forcing a lattice-like finite difference for the time derivative, and that this is actually by an imaginary time displacement. This is similar to the $+\imath\epsilon$ prescription in quantum field theory where operations more naturally take place in Euclidean space and must be Wick rotated back to the Minkowski picture. Note also that the noncommutativity of functions and the time direction generates the exterior derivative in the sense

$$(14) \quad [dt, f] = \imath\lambda df, \quad \forall f$$

which is a typical feature of many noncommutative geometries but has no classical analogue.

Next, at least formally, we have eigenfunctions of the ∂^μ given by

$$(15) \quad \psi_{k,\omega} = e^{\imath kx} e^{\imath\omega t}, \quad \psi_{k,\omega} \psi_{k',\omega'} = \psi_{k+e^{-\lambda\omega}k', \omega+\omega'}, \quad (\psi_{k,\omega})^{-1} = \psi_{-ke^{\lambda\omega}, -\omega}.$$

where we also show the product and inversion of such functions. We see from the latter that the Fourier dual or momentum space is the nonAbelian Lie group $\mathbb{R} \ltimes_\lambda \mathbb{R}^3$. The invariant integration is the usual one on normal ordered functions and hence, allowing for the required ordering, the Fourier transform is given by

$$(16) \quad \begin{aligned} \mathcal{F}(: f(x, t) :)(k, \omega) &= \int \psi_{k,\omega} : f(x, t) := \int dx dt e^{\imath k \cdot x} e^{\imath\omega t} f(e^{-\lambda\omega} x, t) \\ &= e^{\lambda\omega} \mathcal{F}_{\text{usual}}(f)(e^{\lambda\omega} k, \omega), \end{aligned}$$

i.e. reduces to a usual Fourier transform. We also compute the scalar wave operator from $\star d \star d$ and obtain the usual form $(\partial^0)^2 - \sum_i (\partial^i)^2$, which now has massless modes given by plane waves with

$$(17) \quad 2\lambda^2 (\cosh(\lambda\omega) - 1) - k \cdot k e^{\lambda\omega} = 0.$$

This is a straightforward application of the Fourier theory on nonAbelian enveloping algebras introduced in [1].

More details and, in particular, a physical analysis, appeared in [13]. Critically, one has to make a postulate for how the mathematics shall be related to experimental numbers. Here, given the solvable Lie algebra structure, we proposed that expressions shall be identified only when normal ordered. In effect, one measures t first in any experiment. Under such an assumption one can analyse the wave-velocity of the above plane waves and argue that the dispersion relation has the classical form. Both of these steps are needed for any meaning to predictions from the theory. We can then find for the massless wave speed:

$$(18) \quad |d\omega dk| = e^{-\lambda\omega}$$

in units where 1 is the usual speed of light. We assumed that light propagation has the same features as our analysis for massless fields, in which case the physical prediction is that the speed of light depends on energy.

One may then, for example, plug in numbers from gamma-ray burst data as follows. These gamma-ray bursts have been shown in some cases to travel cosmological distances before arriving on Earth, and have a spread of frequencies from 0.1-100 MeV in energy terms. According to the above, the relative time delay Δ_t on travelling distance L for frequencies $\omega, \omega + \Delta_\omega$ is

$$(19) \quad \Delta_t \sim \lambda \Delta_\omega L c \sim 10^{-44} \text{s} \times 100 \text{MeV} \times 10^{10} \text{y} \sim 1 \text{ms}$$

where we put in the worst case for λ , namely the Planck time. We see that arrival times would be spread by the order of milliseconds, which is in principle observable! To observe it would need a statistical analysis of many gamma-ray burst events, to look for an effect that was proportional to distance travelled (since little is known about the initial creation profile of any one burst). This in turn would require accumulation of distance-data for each event by astronomers, such as has been achieved in some cases by coordination between the (now lost) BEPPO-SAX satellite to detect the gamma-ray burst and the Hubble telescope to lock in on the host galaxy during the afterglow period. With the design and implementation of such experiments and statistical analysis, we see that one might in principle observe the effect even if it originates in quantum gravity.

Let us mention finally that there are many other effects of noncommutative space-time, some of which might be measured in earthbound experiments. For example, the LIGO/VIRGO gravitational wave interferometer project, although intended to detect gravitational waves, could also detect the above variable speed of light effect; a detailed theoretical model has yet to be built, but some initial speculations are in [14]. Similarly, reversal of momentum in our theory is done by group inversion, which means $(k, \omega) \rightarrow (-ke^{\lambda\omega}, -\omega)$, a modification perhaps detectable as CPT violation in neutral Kaon resonances[13]. The problem of interpretation in scattering theory is still open, however: what is the meaning of nonAbelian momentum and how might one detect it? Let us not forget also that the usual Lorentz and Poincaré group covariance is modified to a certain quantum group. It contains the usual $U(so_{1,3})$ as a sub-Hopf algebra but acting in a modified non-linear way, which means that special relativity effects are slightly modified. This is another source of potential observability. In short, we have indicated the reasons for Claim (B) above, but much needs to be done by way of physical interpretation and experimental design.

Finally, we comment on the rest of the geometry. The exterior algebra and cohomology holds no surprises (the latter is trivial). Indeed, the space is geometrically

as trivial as $\mathbb{R}^{1,3}$. This is consistent with our view that the above predictions have nothing to do with gravity, it is an independent effect. Thus, the curvature in the Maxwell theory of a gauge field $A = A^\mu dx_\mu$ is

$$(20) \quad F = dA = \partial^\mu A^\nu dx_\mu \wedge dx_\nu$$

and its components have the usual antisymmetric form because the basic forms anticommute as usual. Because the Hodge \star operations on them are also as usual when we keep all differentials to the right, and because the partial derivatives commute, the Maxwell operator $\star d \star d$ on 1-forms has the same form as the usual one, namely the scalar wave operator as above if we take A in Lorentz gauge $\partial^\mu A_\mu = 0$. This is why Maxwell light propagation is as in the scalar field case as assumed above. If we take a static electric source $J = \rho(x)dt$ then the scalar potential and electric flux are as usual, since the spatial derivatives are as usual. Magnetostatic solutions likewise have the same form. Mixed equations with time dependence have the usual form but with ∂^0 for the time derivative. The $U(1)$ -Yang-Mills theory appears more complicated but has similar features to the Maxwell one. Now the curvature is

$$(21) \quad F = (1 + \imath \lambda A^0) dA + [A^0, A^i] dt \wedge dx_i + 12[A^i, A^j] dx_i \wedge dx_j$$

where the extra terms are from $A \wedge A$ using the relations (14) between functions and 1-forms. A gauge transformation is

$$(22) \quad A^i \mapsto u A^i u^{-1} + u(1 + \imath \lambda A^0) \partial^i u^{-1}, \quad A^0 \mapsto u A^0 u^{-1} + u(1 + \imath \lambda A^0) \partial^0 u^{-1}.$$

The Dirac operator and spinor theory requires more machinery and has not yet been worked out in any meaningful (not ad-hoc) manner. Likewise, quantum field theory on the noncommutative Minkowski space is possible, starting with the Fourier transform above, but has not been fully worked out.

2.2. Angular momentum space and fuzzy spheres. Next we look at angular momentum operators but now regarded in a reversed role as noncommutative position space, which means the Lie algebra su_2 with relations

$$(23) \quad [x_i, x_j] = \imath \lambda \epsilon_{ijk} x_k.$$

One may add a commutative time coordinate if desired, but the first remarkable discovery is that this is not required: when one makes the analysis of differential calculi there is only one natural choice and it is already four, not three dimensional! The algebra itself needs no introduction, but some aspects have been studied under the heading ‘fuzzy spheres’. More precisely, these are finite-dimensional matrix algebras viewed as the image of (23) in a fixed spin representation, in which case one is seeing effectively the quotient where the Casimir $x \cdot x$ is equal to a constant. We are not taking this point of view here but working directly with the infinite-dimensional coordinate algebra (23) itself. This model of noncommutative geometry appeared recently in [15] and we give only a brief synopsis of a few aspects. First of all, the reader may ask about the Euclidean group invariance. This is preserved, but again as a quantum group

$$(24) \quad U(su_2) \ltimes \mathbb{C}[SU_2]$$

where SU_2 is a curved momentum space (as promised above). This is an example of a Drinfeld quantum double as well as a partially trivial bicrossproduct. As $\lambda \rightarrow \infty$ it becomes an S^3 of infinite radius, i.e. flat \mathbb{R}^3 acting by usual translations as it

should. The $U(su_2)$ acts by the adjoint action which becomes usual rotations in the limit. We see that nontrivial quantum groups arise in very basic physics wherever noncommutative operators obeying the angular momentum relations are present.

For the differential geometry, we take $\rho(x_i) = \lambda 2\sigma_i$ the usual Pauli-matrix representation and the basic 1-forms $\Lambda^1 = M_2$, the space of 2×2 matrices since the image of ρ in this case is everything. Then

$$(25) \quad dx_i = \lambda 2\sigma_i, \quad (dx_i)x_j - x_j dx_i = \imath \lambda 2\epsilon_{ijk} dx_k + \lambda 4\delta_{ij} e_0, \quad e_0 x_i - x_i e_0 = \lambda dx_i,$$

where e_0 is the 2×2 identity matrix which, together with the Pauli matrices σ_i completes the basis of basic 1-forms. It provides a natural time direction, even though there is no time coordinate. Indeed, the first cohomology is nontrivial and spanned by e_0 , i.e. it is a closed 1-form which is not d of anything (it is denoted by θ in [15]). Nevertheless, like dt in the previous section, we see from (25) that e_0 generates the exterior derivative by commutator,

$$(26) \quad [e_0, f] = \lambda df, \quad \forall f \in U(su_2).$$

The partial derivatives defined by

$$df = (\partial^i f) dx_i + (\partial^0 f) e_0$$

for all f are hard to write down explicitly; they are given in [15]. Nevertheless, the formal group elements

$$(27) \quad \psi_k = e^{\imath k \cdot x}$$

are the plane waves and eigenfunctions for the partial derivatives. For the same reasons as in our previous model in Section 2.1, the scalar Laplacian comes out as $(\partial^0)^2 - \sum_i (\partial^i)^2$ when we take a local Minkowski metric. Its value on plane waves is

$$(28) \quad 1\lambda^2 ((\cos(\lambda|k|2) - 1)^2 + 4\sin^2(\lambda|k|2)).$$

The Maxwell theory may likewise be worked out and has similarities with the usual one due to the fourth dimension provided by e_0 , except that we will only have static solutions since we have no time variable. This time the coordinates are fully ‘tangled up’ by the relations (23) and solutions are rather hard to write down explicitly. One solution, for a uniform electric charge density and spherical boundary conditions at infinity (i.e. constructed as a series of concentric shells) has timelike source $J = e_0$, scalar potential $x \cdot x$ and electric field proportional to x , i.e. radial, see [15]. There is similarly a magnetic solution for a uniform current density. Non-uniform solutions have yet to be worked out due only to their algebraic complexity. The Dirac operator is known and given in [15] also, as are coherent states in which the noncommutative coordinates behave as close as possible to classical. The explicit form of ∂^0 is also interesting and takes the form

$$(29) \quad \partial^0 f = \lambda 8 \sum_i (\partial^i)^2 f + O(\lambda^2)$$

which is the free particle Hamiltonian to lowest order. This is a general feature of many noncommutative algebras, that there is an extra cotangent direction e_0 induced by the noncommutative geometry as generating d by commutator, and one could even say that this is the ‘origin of time evolution’ if one defines the corresponding energy as ∂^0 and asks that it be $\partial\partial t$ in the algebra with a variable t adjoined. Details of such a philosophy will appear elsewhere.

Finally, we note that there are a couple of physical models in which this kind of noncommutative space could appear naturally. One is $2+1$ quantum gravity in a Euclidean version based on an $iso(3)$ -Chern-Simons theory. There one finds[16] that the quantum states have a quantum group symmetry, namely the double (24), suggesting that our above model should provide a description of the relevant effective geometry. The other is with a certain form of ansatz for matrix models in string theory, under which the theory reduces to one on a fuzzy sphere. On the other hand, the problem of formulating the physical consequences of the above noncommutative geometry is independent of the underlying theory of which it may be an effective model.

2.3. θ -space and the Heisenberg algebra. The first proposal for spacetime was probably made by Snyder[17] in the 1940's even before the modern machinery of noncommutative geometry, and took the form

$$(30) \quad [x_\mu, x_\nu] = i\theta_{\mu\nu}, \quad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{C}$$

where $\theta_{\mu\nu}$ were operators with further properties arranged in such a way as to preserve Lorentz covariance. More recently such algebras have been revived by string theorists with θ now a number and called noncommutative ' θ -space'. This is no longer any kind of noncommutative Euclidean or Minkowski space since it does not appear to have any (pseudo)orthogonal group or quantum group appropriate to that. Rather, it is just the usual Heisenberg algebra of quantum mechanics under another context and has a symplectic character. It can also be viewed as a noncommutative torus in an unexponentiated form. For our present treatment we assume that the space is $2n$ -dimensional and $\theta_{\mu\nu}$ nondegenerate. We take the latter in normal form and replaced by a single central variable, say $t = x_0$, i.e. we take

$$(31) \quad [x_i, x_j] = 0, \quad [x_i, x_{-j}] = i\lambda\delta_{ij}t, \quad [x_{-i}, x_{-j}] = 0$$

in terms of new variables grouped as positive and negative index. This is now of our enveloping algebra form generated by a $2n+1$ -dimensional Heisenberg Lie algebra. We will also give a different treatment of (30) by twisting theory in Section 5.

For the calculus we take the standard representation

$$(32) \quad \rho(x_i) = i\lambda \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_{-i}) = i\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_i^t \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(t) = i\lambda \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where e_i is a row vector with 1 in the i -th position and e_i^t is its transpose. The general construction then gives the basic 1-forms dx_0, dx_i, dx_{-i} as certain matrices. They span the image of ρ since the Heisenberg Lie algebra exponentiates to a similar form in this representation. We obtain

$$(33) \quad (dx_{-i})x_{\pm j} = x_{\pm j}dx_{-i}, \quad (dx_i)x_j = x_jdx_i, \quad (dx_i)x_{-j} - x_{-j}dx_i = i\lambda\delta_{ij}dt$$

and dt central. The partial derivatives defined by

$$df = (\partial^i f)dx_i + (\partial^{-i} f)dx_{-i} + (\partial^0 f)dt$$

then turn out to be just the usual derivatives on functions provided these are normal ordered with all x_{-i} to the left of all x_j . This is even simpler than our first example above.

The plane wave eigenfunctions of the partial derivatives are the group elements

(34)

$$\psi_{k_-, k_+, \omega} = e^{ik_- \cdot x_-} e^{ik_+ \cdot x_+} e^{i\omega t}, \quad \psi_{k_-, k_+, \omega} \psi_{k'_-, k'_+, \omega'} = \psi_{k_- + k'_-, k_+ + k'_+, \omega + \omega' + \lambda k_+ \cdot k'_-}$$

for the Heisenberg group. Integration is the usual one on normal ordered functions and hence, allowing for this, the Fourier transform is given by

$$\begin{aligned} \mathcal{F}(: f :)(k_-, k_+, \omega) &= \int \psi_{k_-, k_+, \omega} : f : \\ &= \int dx_- dx_+ dt e^{ik_- \cdot x_-} e^{ik_+ \cdot x_+} e^{i\omega t} f(x_- - \lambda k_+ t, x_+, t) \\ (35) \quad &= \mathcal{F}_{\text{usual}}(f)(k_-, k_+, \omega + \lambda k_- \cdot k_+), \end{aligned}$$

i.e. reduces to a usual Fourier transform.

For the Laplacian, because the algebra does not have a Euclidean covariance it is not very natural to take the usual metric and Hodge \star operator on forms, but if one does this one would have the usual $(\partial^0)^2 - (\partial^-)^2 - (\partial^+)^2$ etc. Since the derivatives are the usual ones on normal ordered expressions, in momentum space it becomes $\omega^2 - k_-^2 - k_+^2$, etc. without any modifications. The same applies in the Maxwell theory; the only subtlety is to keep all expressions normal ordered. On the other hand it is not clear how logical such a metric is since there is no relevant background symmetry of orthogonal type. Probably more natural is a Laplacian like $(\partial^0)^2 - \partial^- \cdot \partial^+$. On the other hand, the $U(1)$ -Yang-Mills theory, as with the Chern-Simons theory if one wants it, involves commutation relations between functions and forms and begins to show a difference.

Finally, if we want the original (30) we should quotient by $t = 1$. Then the calculus is also quotiented by $dt = 0$ and we see that the calculus becomes totally commutative in the sense $[dx_\mu, x_\nu] = 0$ as per the classical case. This is the case relevant to the string theory literature where, indeed, one finds that field theory etc. at an algebraic level has just the same form as classically. This is also the conclusion in another approach based on symmetric categories[18]. Although a bit trivial noncommutative-geometrically, the model is still physically interesting and we refer to the physics literature[19]. As well as D-branes, there are applications to the physics of motion in background fields and the quantum Hall effect. Moreover, the situation changes when one considers nonAbelian gauge theory and/or questions of analysis, where there appear instantons. From the quantum groups point of view one should consider frame bundles and spin connections with a frame group or quantum group of symplectic type using the formalism of[7]. One can also use the automorphisms provided by the quantum double of the Heisenberg algebra. These are some topics for further development of this model.

3. QUANTUM $U(1)$ -YANG-MILLS THEORY ON THE $\mathbb{Z}_2 \times \mathbb{Z}_2$ LATTICE

In this section we move to a different class of examples of noncommutative geometry, where the algebra is that of functions on a group lattice. It is important that the machinery is identical to the one above, just applied to a different algebra and a corresponding analysis of its differential calculus. In this sense it is part of one noncommutative ‘universe’ in a functorial sense and not an ad-hoc construction specific to lattices. In this case we gain meaningful answers even for a finite lattice. In usual lattice theory this would make no sense because constructions are justified

only in the limit of zero lattice spacing, other aspects are errors; in noncommutative geometry the finite lattice or finite group is an exact geometry in its own right.

Specifically, we look at functions $\mathbb{C}[G]$ on a finite group G . In this case the invariant calculi are described by ad-stable subsets $\mathcal{C} \subset G$ not containing the identity. The elements of the subset are the allowed ‘directions’ by which we may move by right translation from one point to another on the group. Hence the elements of \mathcal{C} label the basis of invariant 1-forms $\{e_a\}$. The differentials are

$$(36) \quad \Omega^1 = \mathbb{C}[G].\mathbb{C}\mathcal{C}, \quad df = \sum_{a \in \mathcal{C}} (\partial^a f) e_a, \quad \partial^a = R_a - \text{id}, \quad e_a f = R_a(f) e_a,$$

where $R_a(f) = f((\)a)$ is right translation. There is a standard construction for the wedge product of basic forms as well. This set up is an immediate corollary of the analysis of [5] but has been emphasised by many authors, such as [20] or more recently [21][22]. Moreover, one can take a metric $\delta_{a,b-1}$ in the $\{e_a\}$ basis, which leads to a canonical Hodge \star operation if the exterior algebra is finite dimensional. Then one may proceed to Maxwell and Yang-Mills theory as well as gravity.

We now demonstrate some of these ideas in the simplest case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. It is a baby version of the treatment already given for S_3 in [21] but we take it further to the complete quantum theory in a path integral approach. We take $\mathcal{C} = \{x = (1, 0), y = (0, 1)\}$ as the two allowed directions from each point, i.e. our spacetime consists of a square with the allowed directions being edges. The corresponding basic 1-forms are e_x, e_y and we have

$$df = (\partial^x f) e_x + (\partial^y f) e_y, \quad de_x = de_y = 0, \quad e_x^2 = e_y^2 = \{e_x, e_y\} = 0.$$

The top form is $e_x \wedge e_y$ and the Hodge \star is

$$\star 1 = e_x \wedge e_y, \quad \star e_x = e_y, \quad \star e_y = -e_x, \quad \star(e_x \wedge e_y) = 1.$$

We write a $U(1)$ gauge field as a 1-form $A = A^x e_x + A^y e_y$. Its curvature

$$F = dA + A \wedge A = F^{xy} e_x \wedge e_y, \quad F^{xy} = \partial^x A^y - \partial^y A^x + A^x R_x A^y$$

is covariant as $F \mapsto u F u^{-1}$ under

$$A \mapsto u A u^{-1} + u d u^{-1}, \quad A^a \mapsto u R_a(u) A^a + u \partial^a u^{-1}$$

for any unitary u (any function of modulus 1). We specify also the reality condition $A^* = A$ where the basic forms are self-adjoint in the sense $e_i^* = e_i$. This translates in terms of components as

$$(37) \quad \bar{A}^a = R_a A^a, \quad \bar{F}^{xy} = -R_{xy}(F^{xy})$$

under complex conjugation and implies that $F^* = F$. Such reality should also be imposed in the examples in the previous section if one wants to discuss Lagrangians. Finally, we change variables by

$$(38) \quad A_x + 1 = \lambda_x e^{i\theta_x}, \quad A_y + 1 = \lambda_y e^{i\theta_y},$$

where the $\lambda_x, \lambda_y \geq 0$ are real and θ_x, θ_y are angles. The reality condition means in our case that a connection is determined by real numbers

$$(39) \quad \lambda_1 = \lambda_x(x), \quad \lambda_2 = \lambda_y(x), \quad \lambda_3 = \lambda_x(y), \quad \lambda_4 = \lambda_y(y)$$

and similarly $\theta_1, \dots, \theta_4$.

With these ingredients the Yang-Mills Lagrangian $\mathcal{L}e_x \wedge e_y = -12F^* \wedge \star F$, along the same lines as for any finite group, takes the form (discarding total derivatives),

$$(40) \quad \mathcal{L} = 12|F^{xy}|^2 = 12(\lambda_x^2 \partial^x \lambda_y^2 + \lambda_y^2 \partial^y \lambda_x^2) + \lambda_x^2 \lambda_y^2 - \lambda_x \lambda_y R_y(\lambda_x) R_x(\lambda_y) w_1$$

where the Wilson loop

$$(41) \quad w_1 = \Re e^{i\theta_x} e^{iR_x \theta_y} e^{-iR_y \theta_x} e^{-i\theta_y}$$

is the real part of the holonomy around the square where we displace by x , then by y , then back by x and then back by y , as explained in [21]. When we sum over all points on the lattice, we have the action

$$(42) \quad \mathcal{S} = \sum \mathcal{L} = (\lambda_1^2 + \lambda_3^2)(\lambda_2^2 + \lambda_4^2) - 4\lambda_1 \lambda_2 \lambda_3 \lambda_4 \cos(\theta_1 - \theta_2 + \theta_3 - \theta_4).$$

We now quantise this theory in a path integral approach. We ‘factorise’ the partition function into one where we first hold the λ_i variables fixed and do the θ_i integrals, and then the λ_i integrals. The first step is therefore a lattice $U(1)$ -type theory with fixed λ_i . The latter are somewhat like a background choice of ‘lengths’ associated to our allowed edges. We refer to [21] for a discussion of the interpretation. We assume the usual measure on the gauge fields before making the polar transformation, then

$$(43) \quad \mathcal{Z} = \int d^4 \lambda d^4 \theta \lambda_1 \lambda_2 \lambda_3 \lambda_4 e^{-\alpha \mathcal{S}} = \int_0^\infty d^4 \lambda \lambda_1 \lambda_2 \lambda_3 \lambda_4 e^{-\alpha(\lambda_1^2 + \lambda_3^2)(\lambda_2^2 + \lambda_4^2)} \mathcal{Z}_\lambda$$

where $\alpha > 0$ is a coupling constant and

$$(44) \quad \begin{aligned} \mathcal{Z}_\lambda &= \int_0^{2\pi} d^4 \theta e^{4\alpha \lambda_1 \lambda_2 \lambda_3 \lambda_4 \cos(\theta_1 - \theta_2 + \theta_3 - \theta_4)} \\ &= \int_0^{2\pi} d\theta_2 d\theta_3 d\theta_4 \int_{-\theta_2 + \theta_3 - \theta_4}^{2\pi - \theta_2 + \theta_3 - \theta_4} d\theta e^{\beta \cos(\theta)} \\ &= (2\pi)^3 \int_0^{2\pi} d\theta e^{\beta \cos(\theta)} = (2\pi)^4 I_0(\beta) \end{aligned}$$

where $\beta = 4\alpha \lambda_1 \lambda_2 \lambda_3 \lambda_4$ and I_0 is a Bessel function, and we changed variables to $\theta = \theta_1 - \theta_2 + \theta_3 - \theta_4$. For the expectation values of Wilson loops in this $U(1)$ part of the theory we take the real part of the holonomy $w_n = \cos(n\theta)$ along a loop that winds around our square n times. Then similarly to the above, we have

$$(45) \quad \langle w_n \rangle_\lambda = \langle \cos(n\theta) \rangle_\lambda = I_n(\beta) I_0(\beta).$$

In other words usual Fourier transform on a circle becomes under quantization a Bessel transform,

$$\langle \text{Fourier Transform} \rangle_\lambda \cdot \mathcal{Z}_\lambda = \text{Bessel transform}$$

$$\langle \sum_n a_n \cos(n\theta) \rangle_\lambda \cdot \mathcal{Z}_\lambda = (2\pi)^4 \sum_n a_n I_n(\beta)$$

as the effect of taking the vacuum expectation value. The left hand side here is the unnormalised expectation value. The same results apply in a ‘Minkowski’ theory where α is replaced by $i\alpha$, with Bessel J functions instead. Of course, the appearance of Bessel functions is endemic to lattice theory; here it is not an approximation error but a clean feature of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ theory.

At the other extreme, we consider the reverse order in which the θ_i integrals are deferred. We change variables by

$$\lambda_1 = 12\sqrt{x}(a + \sqrt{2-a^2}), \quad \lambda_2 = 12\sqrt{y}(b + \sqrt{2-b^2})$$

$$\lambda_3 = 12\sqrt{x}(-a + \sqrt{2-a^2}), \quad \lambda_4 = 12\sqrt{y}(-b + \sqrt{2-b^2})$$

where $a, b \in [-1, 1]$ and $x, y > 0$. Then

$$\lambda_1^2 + \lambda_3^2 = x, \quad \lambda_2^2 + \lambda_4^2 = y, \quad \lambda_1\lambda_2\lambda_3\lambda_4 = 14xy(1-a^2)(1-b^2)$$

$$\begin{aligned} \mathcal{Z} &= (2\pi)^3 \int_0^{2\pi} d\theta \mathcal{Z}_\theta \\ \mathcal{Z}_\theta &= 116 \int_0^\infty dx dy \int_0^1 da db e^{-\alpha xy(1-\cos(\theta)(1-a^2)(1-b^2))} xy(1-a^2)(1-b^2) \sqrt{(2-a^2)(2-b^2)}. \end{aligned}$$

Only the product $z = xy$ is relevant here and moving to this and the ratio x/y as variables, the latter gives a logarithmically divergent constant factor which we discard, leaving the z -integral, which we do. We let

$$A(a, b) = (1-a^2)(1-b^2).$$

Then up to an overall factor

$$\begin{aligned} \mathcal{Z}_\theta &= \int_0^\infty dz \int_0^1 da db A z e^{-\alpha z(1-\cos(\theta)A)} \sqrt{(2-a^2)(2-b^2)} \\ (46) \quad &= 1\alpha^2 \int_0^1 da db A(1-\cos(\theta)A)^2 \sqrt{(2-a^2)(2-b^2)}. \end{aligned}$$

This integral is convergent for all $\theta \neq 0$. The unnormalised expectation of the ‘scale Wilson loop’ is similarly

$$\langle \lambda_1 \lambda_2 \lambda_3 \lambda_4 \rangle_\theta \cdot \mathcal{Z}_\theta = 12\alpha^3 \int_0^1 da db A^2(1-\cos(\theta)A)^3 \sqrt{(2-a^2)(2-b^2)}$$

which is again convergent. If $\theta = 0$ we have divergent integrals but can regularise the theory by, for instance, doing the a -integrals from $\epsilon > 0$, giving $1/\epsilon^2$ and $1/\epsilon^4$ divergences respectively. We plot $\langle \lambda_1 \lambda_2 \lambda_3 \lambda_4 \rangle_\theta$ in Figure 2 with $\alpha = 1$. There is a similar but sharper appearance to \mathcal{Z}_θ itself. The point $\theta = \pi/2$ is special and has values

$$\mathcal{Z}_{\pi/2} = 14, \quad \langle \lambda_1 \lambda_2 \lambda_3 \lambda_4 \rangle_{\pi/2} = \pi^2 32,$$

while the minima occur at $\theta = \pi$ with small positive value.

Finally, we look at the full theory. From our first point of view, for the expectation of the $U(1)$ -Wilson loops in the full theory, we need integrals

$$\begin{aligned} &\langle (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^{m w_n} \rangle \cdot \mathcal{Z} \\ &= (2\pi)^4 \int_0^\infty d^4 \lambda (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^{m+1} e^{-\alpha(\lambda_1^2 + \lambda_3^2)(\lambda_2^2 + \lambda_4^2)} I_n(4\alpha \lambda_1 \lambda_2 \lambda_3 \lambda_4). \end{aligned}$$

We change the λ_i variables to z, a, b and discard a log divergent factor as before. Then up to an overall constant,

$$(47) \quad \langle (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^{m w_n} \rangle \cdot \mathcal{Z} = \int_0^\infty dz \int_0^1 da db z^{m+1} A^{m+1} e^{-\alpha z} I_n(\alpha z A) 4^{m+1} \sqrt{(2-a^2)(2-b^2)}.$$

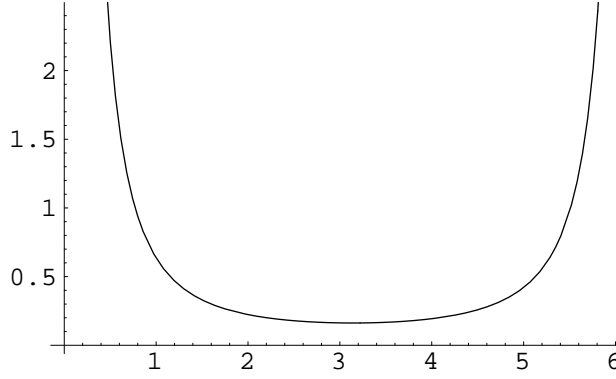


FIGURE 2. Vacuum expectation values $\langle \lambda_1 \lambda_2 \lambda_3 \lambda_4 \rangle_\theta$ plotted for $\theta \in [0, 2\pi]$ and $\alpha = 1$

For example, we have

$$\begin{aligned} \mathcal{Z} &= 14\alpha^2 \int_0^1 da db A(1-A^2)^{32} \sqrt{(2-a^2)(2-b^2)} \\ \langle \lambda_1 \lambda_2 \lambda_3 \lambda_4 \rangle \cdot \mathcal{Z} &= 116\alpha^3 \int_0^1 da db A(2+A^2)(1-A^2)^{52} \sqrt{(2-a^2)(2-b^2)} \\ \langle w_1 \rangle \cdot \mathcal{Z} &= 14\alpha^2 \int_0^1 da db A^2(1-A^2)^{32} \sqrt{(2-a^2)(2-b^2)} \end{aligned}$$

These are all divergent, with $Z \sim 1/\alpha^2 \epsilon$ and so on. The physical reason is the singular contribution from configurations in the functional integral where $\theta = 0$ as seen above. Moreover, if we regularise by doing the a -integral (say) from $\epsilon > 0$ then

$$(48) \quad \langle (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^m \rangle \sim 1\alpha^m \epsilon^{2m} \sim \langle (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^m w_1 \rangle.$$

In particular, we have a finite answer $\langle w_1 \rangle = 1$.

In summary, for the full theory we see that some of the physical vacuum expectation values connected with the phase part of the noncommutative gauge theory, such as the Wilson loop $\langle w_1 \rangle$, are finite but not necessarily interesting. However, if one makes α itself a divergent function of the regulator (such as $\alpha = 1/\epsilon^2$), one can render all the $\langle (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^m \rangle$ and $\langle (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^m w_1 \rangle$ finite as well and indeed one obtains nontrivial answers. A systematic treatment will not be attempted here (one should choose the regulator more physically, among other things) but it does appear that the theory is largely renormalisable.

4. CLIFFORD ALGEBRAS AS NONCOMMUTATIVE SPACES

An idea for ‘quantisation’ in physics is the Moyal product where we work with the same vector space of functions on \mathbb{R}^n but modify the product to a noncommutative one. Exactly the same multiplication-alteration idea can be used to construct Clifford algebras on the vector space $\mathbb{C}[(\mathbb{Z}_2)^n]$ but modifying its product. The formulae are similar but instead of real-valued vectors $x \in \mathbb{R}^n$ we work with \mathbb{Z}_2 -valued vectors, so that Clifford algebras are in a precise sense discrete ‘quantisations’ of the $(\mathbb{Z}_2)^n$ lattice. We will explain this a bit more from the Drinfeld-twist point of

view in [23][24], but for the present purposes the general idea of building Clifford algebras by multiplication-alteration factors is well known. It is mentioned in [25] for example, as well as in the mathematics literature where Clifford algebras were constructed as twisted $(\mathbb{Z}_2)^n$ group rings in [26]. On the other hand, our point of view tells us immediately how to do the differential calculus etc. on such spaces and that one has the same cohomology, eigenvalues of the Laplace operator etc. as the ‘classical’ untwisted case. Before doing this, Section 4.1 finishes the differential geometry for the ‘classical’ case $(\mathbb{Z}_2)^n$ as in Section 3, but now for general n .

4.1. Cohomology and Maxwell theory on $(\mathbb{Z}_2)^n$. We use the setting for finite groups as in Section 3, but now functions are on $(\mathbb{Z}_2)^n$. For \mathcal{C} we take the n directions where we step +1 in each of the \mathbb{Z}_2 directions, leaving the others unchanged. The differentials are

$$(49) \quad (\partial^a f)(x) = f(x + (0, \dots, 1, \dots, 0)) - f(x),$$

where 1 is in the a -th position. We denote by e_a the basic 1-forms, so that

$$(50) \quad df = \sum_{a=1}^n (\partial^a f) e_a, \quad e_a f = R_a(f) e_a, \quad e_a^2 = 0, \quad \{e_a, e_b\} = 0, \quad de_a = 0$$

where $R_a(f) = f(x + (0, \dots, 1, \dots, 0))$ in the a -th place. Because of the cyclic nature of the group \mathbb{Z}_2 it is easy to see that no function can obey $\partial^1 f = 1$ (or similarly in any other direction). From this kind of argument one can find that the noncommutative de Rham cohomology (the closed forms modulo the exact ones) is represented by the grassmann algebra Λ of basic forms generated by the $\{e_a\}$ with the anticommutativity relations in (50),

$$(51) \quad H^*(\mathbb{C}[(\mathbb{Z}_2)^n]) = \Lambda.$$

Here $H^1 = \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_n$ has the same form as for a classical torus $S^1 \times \dots \times S^1$ and for the same reason (and the same holds for $(\mathbb{Z}_m)^n$ for all m). In particular, the top form is $e_1 \cdots e_n$ and is not exact. Moreover, because the exterior algebra among the basic forms is just the usual one, the Hodge \star is the usual one given by the totally antisymmetric epsilon tensor. We take the Euclidean metric δ_{ab} on the basic forms. Then the spin zero wave operator is

$$(52) \quad \square = 14 \star d \star d = 14 \sum_a \partial^a \partial^a = -12 \sum_a \partial^a.$$

The plane wave eigenfunctions of the partial derivatives ∂^a are

$$(53) \quad \psi_k(x) = (-1)^{k \cdot x}, \quad \partial^a \psi_k = -2k_a \psi_k$$

labelled by momenta $k \in (\mathbb{Z}_2)^n$. These diagonalise the wave operator, which has eigenvalue

$$(54) \quad \square \psi_k = |k| \psi_k, \quad |k| = \sum_a k_a = k \cdot k$$

since each $k_a \in \{0, 1\}$. The eigenvalues range from $0, \dots, n$ and there are (nm) eigenfunctions for eigenvalue m . In particular, there are no massless modes other than the constant function 1.

For the spin 1 equation, since the formulae among basic 1-forms are as for \mathbb{R}^n , the Maxwell operator $\star d \star d$ on 1-forms $A = \sum_a A^a e_a$ is given by the above \square on each component function if A is in Lorentz gauge $\partial \cdot A = 0$. Its eigenvalues therefore also range from $0, \dots, n$ according to the degree of all components. The Maxwell

operator on the $(n-1)2^n + 1$ dimensional space of gauge fields A such that $\partial \cdot A = 0$ can be fully diagonalised by the following modes. There are (i) $n2^{n-1}$ modes of the form

$$(55) \quad A = \psi_k e_a$$

of degree $|k| = 0, \dots, n-1$, where $a = 1, \dots, n$ and k is such that its a -component k_a vanishes. In addition there are (ii) $(nm)(m-1)$ modes of degree $m = 2, \dots, n$ of the form

$$(56) \quad A = \psi_k(\mu_1 e_{a_1} + \dots + \mu_m e_{a_m}), \quad \mu_1 + \dots + \mu_m = 0,$$

where the momentum vector k has components $k_{a_1} = \dots = k_{a_m} = 1$ and the rest zero. Adding up the modes of type (ii) gives $n2^{n-1} + 1 - 2^n$, which with the modes of type (i) gives a total of $(n-1)2^n + 1$, which is the required number since the space of A is $n2^n$ -dimensional while the image of d , i.e. the number of exact forms, is $2^n - 1$ -dimensional. Note that the gauge fixing by $\partial \cdot A$ still leaves possible $A \mapsto A + df$ where $\square f = 0$ but then $f = 1$ by the above, and $df = 0$, i.e. the gauge-fixing is fully effective. These eigenfunctions A with degree > 0 are also the allowed sources, being a basis of solutions of $\partial \cdot J = 0$ in the image of the Maxwell wave operator. Hence we can solve fully the Maxwell equations. One could also rework these results with a Minkowski metric in the e_a basis.

4.2. Noncommutative geometry on Clifford algebras. In this section we look at the standard Clifford algebra $\text{Cliff}(n)$ with generators γ_a , $a = 1, \dots, n$ and relations

$$(57) \quad \{\gamma_a, \gamma_b\} = \delta_{a,b}.$$

In [26, 24] this is presented as a twist of the group algebra of $(\mathbb{Z}_2)^n$. This is the momentum group for our classical space $\mathbb{C}[(\mathbb{Z}_2)^n]$ above, with elements the plane waves ψ_k . The ‘quantisation’ procedure consists of changing the product between such basis vectors to a new one

$$(58) \quad \psi_k \bullet \psi_m = F(k, m) \psi_{k+m}, \quad F(k, m) = (-1)^{\sum_{j < i} k_i m_j}$$

where F is a cocycle on the group $(\mathbb{Z}_2)^n$ just because it is bilinear. These relations are such that if we write

$$(59) \quad \psi_a = \psi_{(0, \dots, 1, \dots, 0)}, \quad \psi_a(x) = \begin{cases} -1 & \text{if } x_a = 1 \\ 0 & \text{else} \end{cases},$$

where 1 in the momentum vector is in the a -th position, then

$$\psi_a \bullet \psi_b + \psi_b \bullet \psi_a = \delta_{a,b}$$

so that we can identify $\gamma_a = \psi_a$. We similarly define

$$(60) \quad \gamma_k = \gamma_1^{k_1} \dots \gamma_n^{k_n}$$

for any $k \in (\mathbb{Z}_2)^n$ and identify $\gamma_k = \psi_k$. In this way the whole basis of $\mathbb{C}[(\mathbb{Z}_2)^n]$ is identified with the vector space of the Clifford algebra. This approach makes $\text{Cliff}(n)$ a braided-commutative algebra in a symmetric monoidal category, see [24].

We can similarly identify all normal ordered expressions where the γ_a occur in increasing order from left to right with the identical ‘classical’ expression in terms of the commutative ψ_a . Moreover, because the Clifford algebra is obtained by a cocycle twist, it follows from the general theory in the next section that we may likewise identify the differential calculi. Indeed, the entire exterior algebra

$\Omega(\text{Cliff}(n))$ is a twist by the same cocycle F but now of the exterior algebra of differential forms on $\mathbb{C}[(Z_2)^n]$ from Section 4.1, with the same cohomology and gauge theory. Explicitly,

$$(61) \quad \Omega(\text{Cliff}(n)) = \text{Cliff}(n) \cdot \Lambda,$$

where Λ is our previous grassmann algebra (50) generated by the anticommuting e_a . We have

$$(62) \quad df = \sum_a (\partial^a f) e_a, \quad \partial^a : f :=: \partial^a f :,$$

where $: f :$ is the normal ordered element defined by writing f in terms of the ψ_a and on the right we use the partial derivatives from the previous section. Similarly for the noncommutation relations between functions and 1-forms. Explicitly,

$$(63) \quad \partial^a \gamma_k = -2k_a \gamma_k, \quad e_a \gamma_b = \begin{cases} -\gamma_b e_a & \text{if } a = b \\ \gamma_b e_a & \text{else} \end{cases}.$$

Taking the δ_{ab} metric on the e_a basis as before, and the corresponding Hodge \star etc., we have the Laplace operator $14 \star d \star d$ on plane waves

$$(64) \quad \square \gamma_k = |k| \gamma_k$$

so that the total degree function on the Clifford algebra has a nice geometrical interpretation as the eigenvalue of the Laplacian. Similarly, the diagonalisation of the wave operator in spin 1 has the same form as in Section 4.1. For example, in $\text{Cliff}(2)$ there are five eigenfunctions with $\partial \cdot A = 0$, namely

$$e_1, e_2, \gamma_2 e_1, \gamma_1 e_2, \gamma_1 \gamma_2 (e_1 - e_2).$$

The allowed sources in the Maxwell theory are spanned by the latter three. The corresponding solutions for A and their curvatures are respectively

$$A = \gamma_2 e_1, \gamma_1 e_2, 12\gamma_1 \gamma_2 (e_1 - e_2), \quad F = dA = 2\gamma_2 e_1 \wedge e_2, -2\gamma_1 e_1 \wedge e_2, 2\gamma_1 \gamma_2 e_1 \wedge e_2.$$

One also has a $U(1)$ -Yang-Mills theory on the Clifford algebra, where the curvature is $dA + A \wedge A$ and gauge transform is (2) under an invertible element of the Clifford algebra. The subgroup of such gauge transformations restricted to preserve the generating space $\{\gamma_a\}$ under the Clifford adjoint operation is the Clifford group, giving a geometrical interpretation of that.

5. TWISTING OF DIFFERENTIALS AND THE NONCOMMUTATIVE TORUS A_θ .

In this more technical section we explain the theorem for the systematic twisting of differentials on quantum groups and a small corollary of it which covers our above treatment of differential calculus on Clifford algebras as well as on the noncommutative torus and other spaces. The theory works at the level of a Hopf algebra H but for the purposes of the present article this could just be the coordinate algebra $\mathbb{C}[G]$ of an algebraic group or the group algebra of a discrete group; Hopf algebras are a nice way to treat the continuum and finite theories on an identical footing, as well as allowing one to generalise to quantum groups if one wants q -deformations etc.

A 2-cocycle on the group in our terms is an element $F : H \otimes H \rightarrow \mathbb{C}$ obeying the condition

$$(65) \quad F(b_{(1)} \otimes c_{(1)}) F(a \otimes b_{(2)} c_{(2)}) = F(a_{(1)} \otimes b_{(1)}) F(a_{(2)} b_{(2)} \otimes c), \quad F(1 \otimes a) = \epsilon(a)$$

for all $a, b, c \in H$, where $\Delta a = a_{(1)} \otimes a_{(2)} \in H \otimes H$ is our notation for the coproduct. We will similarly denote $(\text{id} \otimes \Delta)\Delta a = (\Delta \otimes \text{id})\Delta a = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ for the iterated coproduct. On a group manifold, the counit ϵ simply evaluates at the group identity. It is a construction going back to Drinfeld that in this case there is a new Hopf algebra H_F with product

$$(66) \quad a \bullet b = F(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}F^{-1}(a_{(3)} \otimes b_{(3)}), \quad \forall a, b \in H$$

where F^{-1} is the assumed convolution inverse. See [27] for details and proofs. Another fact is that for a bicovariant calculus on a Hopf algebra H , the exterior algebra $\Omega(H)$ is a super-Hopf algebra[29]. Then:

Theorem 5.1. (*S.M. & R. Oeckl [3]*) *Let F be a 2-cocycle on a Hopf algebra H with bicovariant calculus $\Omega(H)$. Extend F to a super-cocycle on $\Omega(H)$ by zero on degree > 0 . Then $\Omega(H_F) = \Omega(H)_F$ is a bicovariant calculus on H_F .*

This took care of how calculi on Hopf algebras themselves behave under twisting. We need a slight but immediate extension of this. Let A be an algebra on which H coacts by an algebra homomorphism $\Delta_L : A \rightarrow H \otimes A$. We will use the notation $\Delta_L a = a^{(\bar{1})} \otimes a^{(\bar{\infty})}$. In the case when H is functions on a group, given any group element we evaluate the H part of the output of Δ_L on the element and have a map $A \rightarrow A$, i.e. a coaction is just the same thing as a group action but expressed in terms of the coordinates on the group. A fundamental theorem [28] is that when we twist H to H_F we must also twist A to a new algebra A_F for it to be H_F -covariant; its product is

$$(67) \quad a \bullet b = F(a^{(\bar{1})} \otimes b^{(\bar{1})})a^{(\bar{\infty})}b^{(\bar{\infty})}, \quad \forall a, b \in A.$$

There is obviously the same theorem in the category of super algebras coacted upon by super-Hopf algebras, which is trivial to spell out (one just puts the signed super-transposition in place of the usual transposition in all constructions). Now suppose that A is covariant under H and is equipped with a covariant differential calculus such that the coaction extends to all of the exterior super-algebra $\Omega(A)$ of (possibly noncommutative) differential forms as a comodule algebra under H . So we can apply (67) to $\Omega(A)$ in place of A . Clearly,

Corollary 5.2. *If $\Omega(A)$ is H -covariant on an H -covariant algebra A and if F is a cocycle on H then $\Omega(A_F) = \Omega(A)_F$ is an H_F -covariant calculus on A_F .*

One could also view $\Omega^1(A)$ as a $\Omega^1(H)$ -super comodule algebra and then using Theorem 5.1 and the super-version of (67) applied to $\Omega(A)$ we see in fact that $\Omega(A_F)$ is an $\Omega(H_F)$ -super comodule algebra.

Since any Hopf algebra coacts covariantly on itself via $\Delta_L = \Delta$, a canonical example is to take $A = H$ with this coaction. Then for any cocycle F we have a new algebra A_F via (67) covariant under H_F obtained via (66). This was the theory behind [23, 24]. Now suppose that $\Omega(H)$ is a bicovariant calculus on H , then it is in particular left covariant. So $\Omega(A) = \Omega(H)$ is left H -covariant as induced by Δ_L on functions. We can therefore apply Corollary 5.2 to obtain $\Omega(A_F)$ also. In fact[5] $\Omega(A) \cong A \otimes \Lambda$ where Λ is the algebra of left-invariant forms generated by $\{e_a\}$ say. The coaction on these forms is trivial and hence when we apply (67) to $\Omega(A)$ we

find simply

$$\begin{aligned}
 e_a \wedge_\bullet e_b &= F(e_a^{(\bar{1})} \otimes e_b^{(\bar{1})}) e_a^{(\bar{\infty})} \wedge e_b^{(\bar{\infty})} = e_a \wedge e_b \\
 a \bullet e_a &= F(a^{(\bar{1})} \otimes e_a^{(\bar{1})}) a^{(\bar{\infty})} e_a^{(\bar{\infty})} = a e_a \\
 (68) \quad e_a \bullet a &= F(e_a^{(\bar{1})} \otimes a^{(\bar{1})}) e_a^{(\bar{\infty})} a^{(\bar{\infty})} = e_a a, \quad \forall a \in A.
 \end{aligned}$$

So the form relations are unchanged in this basis $\{e_a\}$. We get the same result in the super-twisting point of view: the supercoproduct in $\Omega(H)$ looks like $\underline{\Delta}e_a = \Delta_R e_a + 1 \otimes e_a$ where $\Delta_R e_a \in \Lambda^1 \otimes H$ is a certain right coaction. Here Λ^1 is spanned by the $\{e_a\}$. Since the cocycle is trivial on 1 in the sense in (65) and similarly for $F(a \otimes 1)$, and since extended by zero on the e_a , we see that the super version of (67) with $\Omega^1(A) = \Omega^1(H)$ and supercoaction $\underline{\Delta}_L = \underline{\Delta}$ gives the same result that all of these products involving 1-forms are unchanged.

Let us see how all of this works for a discrete group G . Here $H = \mathbb{C}G$ is the group algebra spanned by group elements, with coproduct $\Delta g = g \otimes g$ for all $g \in G$. The equation (65) then reduces to a group cocycle. Yet, because H is cocommutative, the twisting (66) has no effect and $H_F = H$ is unchanged for all F . On the other hand, we can take $A = \mathbb{C}G$ as a left-covariant algebra under $\Delta_L = \Delta$. This time (67) means a new algebra A_F with product

$$(69) \quad g \bullet h = F(g, h)gh, \quad \forall g, h \in G,$$

extended linearly. This is the special case used in Section 4.

We now look at the initial differential structure. To keep the picture simple we assume G is Abelian (this is not necessary). Then under Fourier transform $\mathbb{C}G \cong \mathbb{C}[\hat{G}]$, where \hat{G} is the group of characters (if G is infinite then this will be compact and $\mathbb{C}[\hat{G}]$ is the algebraic coordinate ring). In the compact case we use indeed the classical calculus on this ‘position space’ \hat{G} while in the finite case we use the set-up (36) for a chosen conjugacy class. That is the geometrical picture, but we do not have to actually do the Fourier transform to the coordinate ring picture, we rather work directly with the ‘momentum group’ G dual to the position space. Then

$$(70) \quad \Omega(\mathbb{C}G) = (\mathbb{C}G)\Lambda$$

where Λ here is the usual grassmann algebra of basic forms $\{e_a\}$. The differential and relations have the form

$$(71) \quad dg = \sum_a (\chi_a(g) - 1)ge_a, \quad e_a g = \chi_a(g)ge_a, \quad \forall g \in G$$

and the super-coproduct is

$$(72) \quad \underline{\Delta}g = g \otimes g, \quad \underline{\Delta}e_a = e_a \otimes 1 + 1 \otimes e_a.$$

Here $\{\chi_a\}$ are some subset of characters, the allowed directions in \hat{G} that define the calculus. This is equivalent to our treatment in Section 4.1 for $G = (\mathbb{Z}_2)^n$, with g in the role of plane waves ψ_k .

Next we take $A = \mathbb{C}G = H$ another copy of the same algebra, F a group cocycle and $\Delta_L = \Delta$ to get our twisted version $A_F = (\mathbb{C}G)_F$ as in (69). And we take $\Omega(A) = (\mathbb{C}G)\Lambda = \Omega(H)$ as above. Then from (68) we know that $\Omega((\mathbb{C}G)_F)$ has the identical form to the untwisted case. The only part that changes is the

algebra $(CG)_F$ itself which becomes twisted and typically (depending on F) noncommutative. This is the reason for exactly the same form of calculus on $\text{Cliff}(n)$ in Section 4.2 as for $(\mathbb{Z}_2)^n$.

Now let us cover the celebrated noncommutative torus A_θ in the same way. This has the relations $vu = e^{i\theta}uv$ where θ is an angle. For our initial situation we take $G = \mathbb{Z} \times \mathbb{Z}$ in (71), with free commuting generators u, v . Here the character group is $S^1 \times S^1$ and the calculus is defined by two characters

$$\chi_{1,\phi}(u^m v^n) = e^{i\phi m}, \quad \chi_{2,\phi}(u^m v^n) = e^{i\phi n}$$

with ϕ as a parameter. The only difference is that we rescale d and then take the limit $\phi \rightarrow 0$ of this family of calculi, giving

$$\begin{aligned} d(u^m v^n) &= \lim_{\phi \rightarrow 0} 1\phi ((\chi_{1,\phi}(u^m v^n) - 1)u^m v^n e_1 + (\chi_{2,\phi}(u^m v^n) - 1)u^m v^n e_2) \\ &= u^m v^n (m e_1 + n e_2) \\ e_a u^m v^n &= u^m v^n e_a. \end{aligned}$$

This is nothing but the usual classical differential calculus on $\mathbb{C}[S^1 \times S^1]$, but written algebraically and in momentum space. Next, on $\mathbb{Z} \times \mathbb{Z}$ we take cocycle

$$(73) \quad F(u^m v^n, u^s v^t) = e^{i\theta ns},$$

where θ denotes a fixed parameter. Then the formula (69) gives

$$(74) \quad v \bullet u = e^{i\theta}uv = e^{i\theta}u \bullet v$$

which is noncommutative torus from our algebraic point of view. That its product is a twisting is known to experts from another point of view [31] and also for more general θ -spaces [32] from the twisting point of view in recent work [33, 34]. That one automatically gets the differential geometry as a twist seems to be less well-known. Thus, for the differential structure we obtain

$$(75) \quad du = ue_1, \quad dv = ve_2, \quad e_a u = ue_a, \quad e_a v = ve_a$$

since this is the same as the classical form. Likewise the wedge products are as per the classical form in which the e_a anticommute. Note then that

$$(dv)u = ve_1 u = v \bullet ue_1 = e^{i\theta}u \bullet ve_1 = e^{i\theta}udv$$

$$dv \wedge du = ve_2 \wedge ue_1 = e_2 \wedge v \bullet ue_1 = e^{i\theta}e_2 \wedge u \bullet ve_1 = -e^{i\theta}du \wedge dv$$

which is how the calculus is usually presented, as noncommutation between du, v etc., but we see that it expresses nothing more than the noncommutativity of A_θ itself. This goes some way towards explaining why one can develop Yang-Mills and other geometry on a noncommutative torus [30] so much like on a usual torus, with two commuting derivations as vector fields, etc. For a general noncommutative algebra one does not expect many derivations.

We can just as easily apply this theory to $H = A = \mathbb{C}[\mathbb{R}^n]$. Here H has the linear coproduct $\Delta x_\mu = x_\mu \otimes 1 + 1 \otimes x_\mu$ and on it we take the cocycle

$$(76) \quad F(f \otimes g) = e^{i \sum \theta_{\mu\nu} \partial^\mu \otimes \partial^\nu} (f \otimes g)(0)$$

where we apply differential operators on functions f, g and evaluate at zero. Then the algebra $\mathbb{C}[\mathbb{R}^n]_F$ has product

(77)

$$f \bullet g = \left(\sum_{m=0}^{\infty} \left(\sum \theta_{\mu\nu} \partial^\mu \otimes \partial^\nu \right)^m m!|_0 (f_{(1)} \otimes g_{(1)}) \right) f_{(2)} g_{(2)} = \cdot e^{\sum_{\mu,\nu} \theta_{\mu\nu} \partial^\mu \otimes \partial^\nu} (f \otimes g)$$

where $(\partial^m|_0 f_{(1)}) f_{(2)} = \partial^m f$. This is the Moyal product and this point of view was explored in [3] in the context of generalising the cocycle to a nonAbelian group rather than \mathbb{R}^n . It was picked up and discussed explicitly in [18] as well as by other authors. In particular, the coordinate functions x_μ with the \bullet product obey the algebra (30) studied in Section 2.3. From this point of view the bilinear form that defines the Clifford algebra twisting (58) is like a finite difference matrix. Meanwhile the cochain that similarly twists $(\mathbb{Z}_2)^n$ to the octonions [23] has an additional cubic term that is responsible for their nonassociativity, i.e. more like a discrete ‘Chern-Simons’ theory as remarked there. Note that when F is only a cochain our Corollary 5.2 induces not a usual associative exterior algebra on the octonions but a superquasialgebra in the same sense as the octonions are a quasialgebra (i.e. associative when viewed in a monoidal category).

Further afield, other θ -manifolds have been of interest of late, see [35] and references therein. It is clear that these too are twistings as noncommutative algebras and therefore that one can recover the algebraic side of their noncommutative geometry by twisting the classical geometry as above. It is assumed that M is a classical manifold admitting a group action by a compact Lie group $K \supseteq S^1 \times S^1$. We assume an algebraic description exists (otherwise one needs to do some analysis), so that we have classical coordinate Hopf algebras $\mathbb{C}[K] \rightarrow \mathbb{C}[S^1 \times S^1]$ and a coaction $\mathbb{C}[M] \rightarrow \mathbb{C}[K] \otimes \mathbb{C}[M]$. We suppose that this extends to the classical exterior algebra of M also. The action of the subgroup here corresponds to a coaction $\mathbb{C}[M] \rightarrow \mathbb{C}[S^1 \times S^1] \otimes \mathbb{C}[M]$. Then with the same cocycle (73), we obtain a new algebra $\mathbb{C}[M]_F$ by the comodule algebra twisting theorem (67), and we obtain a differential calculus $\Omega(\mathbb{C}[M]_F)$ on it by Corollary 5.2, and ultimately an entire twisted noncommutative geometry with a parameter θ . Equivalently, we can obviously pull back F as a cocycle $F : \mathbb{C}[K] \otimes \mathbb{C}[K] \rightarrow \mathbb{C}$ and do all of the above with the $\mathbb{C}[K]$ -coaction directly (the result is the same). On the other hand, whereas $\mathbb{C}[S^1 \times S^1]$ does not itself twist as a Hopf algebra, now (66) gives a new Hopf algebra $\mathbb{C}[K]_F$ and this coacts on $\Omega(\mathbb{C}[M]_F)$ by Corollary 5.2. By Theorem 5.1 we also have $\Omega(\mathbb{C}[K]_F)$ and this supercoacts. It seems that these elementary deductions fit with observations from a different point of view (not via the twisting theory) in [35]. We see only an ‘easy’ algebraic part of that theory but it does seem to indicate some useful convergence between that operator theory approach and the quantum groups one.

REFERENCES

- [1] S. Majid. Duality principle and braided geometry. Springer Lec. Notes in Phys. 447:125–144, 1995.
- [2] A. Connes. *Noncommutative Geometry*. Academic Press, 1994.
- [3] S. Majid and R. Oeckl. Twisting of quantum differentials and the Planck scale Hopf algebra. *Commun. Math. Phys.*, 205:617–655, 1999.
- [4] S. Majid. Hopf algebras for physics at the Planck scale. *J. Classical and Quantum Gravity*, 5:1587–1606, 1988.

- [5] S.L. Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Commun. Math. Phys.*, 122:125–170, 1989.
- [6] T. Brzeziński and S. Majid. Quantum group gauge theory on quantum spaces. *Commun. Math. Phys.*, 157:591–638, 1993. Erratum 167:235, 1995.
- [7] S. Majid. Riemannian geometry of quantum groups and finite groups with nonuniversal differentials. *Commun. Math. Phys.*, 225:131–170, 2002.
- [8] F. Ngakeu, S. Majid and D. Lambert. Noncommutative Riemannian geometry of the alternating group A_4 . *J. Geom. Phys.*, 42:259–282, 2002.
- [9] S. Majid. Ricci tensor and Dirac operator on $\mathbb{C}_q[SL_2]$ at roots of unity. *Lett. Math. Phys.*, 63:39–54, 2003.
- [10] S. Majid. On q -Regularization. *Int. J. Mod. Phys. A*, 5:4689–4696, 1990.
- [11] S. Majid and H. Ruegg. Bicrossproduct structure of the κ -Poincaré group and non-commutative geometry. *Phys. Lett. B*, 334:348–354, 1994.
- [12] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy. , q -Deformation of Poincaré algebra. *Phys. Lett. B*, 271:321, 1991.
- [13] G. Amelino-Camelia and S. Majid. Waves on noncommutative spacetime and gamma-ray bursts. *Int. J. Mod. Phys. A*, 15:4301–4323, 2000.
- [14] G. Amelino-Camelia. *Gravity-wave interferometers as quantum-gravity detectors*. *Nature* 398:216, 1999.
- [15] E. Batista and S. Majid. Noncommutative geometry of angular momentum space $U(su_2)$. *J. Math. Phys.*, 44:107–137, 2003.
- [16] B.J. Schroer. Combinatorial quantization of Euclidean gravity in three dimensions, *preprint* math.QA/0006228.
- [17] S. Snyder. Quantized space-time. *Phys. Rev.*, 71:38–41, 1947.
- [18] R. Oeckl. Untwisting noncommutative \mathbb{R}^d and the equivalence of quantum field theories. *Nucl.Phys. B* 581:559–574, 2000.
- [19] N. Seiberg and E. Witten. String theory and noncommutative geometry. *J. High En. Phys.*, 9909:032, 1999.
- [20] K. Bresser, F. Mueller-Hoissen, A. Dimakis, A. Sitarz. Noncommutative geometry of finite groups. *J. Phys. A* 29:2705–2736, 1996.
- [21] S. Majid and E. Raineri. Electromagnetism and gauge theory on the permutation group S_3 . *J. Geom. Phys.*, 44:129–155, 2002.
- [22] S. Majid and T. Schucker. $\mathbb{Z}_2 \times \mathbb{Z}_2$ lattice as a Connes-Lott-quantum group model. *J. Geom. Phys.*, 43:1–26, 2002.
- [23] H. Albuquerque and S. Majid. Quasialgebra structure of the Octonions. *J. Algebra*, 220:188–224, 1999.
- [24] H. Albuquerque and S. Majid. Clifford algebras obtained by twisting of group algebras. *J. Pure Applied Algebra* 171:133–148, 2002.
- [25] G. Dixon. *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*. Kluwer, 1994.
- [26] S. Caenepeel, F. Van Oystaeyen. A note on generalized Clifford algebras and representations. *Comm. in Algebra* 17:93–102, 1989.
- [27] S. Majid. *Foundations of Quantum Group Theory*. Cambridge Univeristy Press, 1995.
- [28] S. Majid. q -Euclidean space and quantum Wick rotation by twisting. *J. Math. Phys.*, 35:5025–5034, 1994.
- [29] T. Brzeziński. Remarks on bicovariant differential calculi and exterior Hopf algebras. *Lett. Math. Phys.*, 27:287, 1993.
- [30] A. Connes and M.A. Rieffel. Yang-Mills for noncommutative two-tori. *AMS Contemp. Math. Series* 62:237–266, 1987.
- [31] M. Rieffel. Deformation quantization for actions of \mathbb{R}^d . *Memoirs AMS*, 106, 1993.
- [32] A. Connes and G. Landi. Noncommutative manifolds, the instanton algebra and isospectral deformations. *Commun. Math. Phys.* 221:141–159, 2001.
- [33] A. Sitarz. Twist and spectral triples for isospectral deformations. *Lett. Math. Phys.* 58: 69–79, 2001.
- [34] J. Varilly. Quantum symmetry groups of noncommutative spheres. *Commun. Math. Phys.* 221:511–523, 2001.
- [35] A. Connes and M. Dubois-Violette. Noncommutative finite-dimensional manifolds, I: Spherical manifolds and related examples. *Commun. Math. Phys.* 230: 539–579, 2002.

SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY, UNIVERSITY OF LONDON, 327 MILE END
RD, LONDON E1 4NS, UK.